Stability of Some Positive Linear Operators on Compact Disk

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Abstract. Recently, Popa and Raşa [18, 19] have shown the (in)stability of some classical operators defined on [0, 1] and found best constant when the positive linear operators are stable in the sense of Hyers-Ulam. In this paper we show Hyers-Ulam (in)stability of complex Bernstein-Schurer operators, complex Kantrovich-Schurer operators and Lorentz operators on compact disk. In the case when the operator is stable in the sense of Hyers and Ulam, we find the infimum of Hyers-Ulam stability constants for respective operators.

1. Introduction

The equation of homomorphism is stable if every "approximate" solution can be approximated by a solution of this equation. The problem of stability of a functional equation was formulated by S.M. Ulam [23] in a conference at Wisconsin University, Madison in 1940: "Given a metric group $(G, ., \rho)$, a number $\varepsilon > 0$ and a mapping $f : G \to G$ which satisfies the inequality $\rho(f(xy), f(x)f(y)) < \varepsilon$ for all $x, y \in G$, does there exist a homomorphism a of G and a constant $k > 0$, depending only on G, such that $\rho(a(x), f(x)) \leq k\varepsilon$ for all $x \in G$?" If the answer is affirmative the equation $a(xy) = a(x)a(y)$ of the homomorphism is called stable; see [5, 10]. The first answer to Ulam's problem was given by D.H. Hyers [9] in 1941 for the Cauchy functional equation in Banach spaces, more precisely he proved: "Let X, Y be Banach spaces, ε a non-negative number, $f : X \to Y$ a function satisfying $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$, then there exists a unique additive function with the property $\|f(x) - a(x)\| \leq \varepsilon$ for all $x \in X$." Due to the question of Ulam and the result of Hyers this type of stability is called today Hyers-Ulam stability of functional equations. A similar problem was formulated and solved earlier by G. Pólya and G. Szegö in [16] for functions defined on the set of positive integers. After Hyers result a large amount of literature was devoted to study Hyers-Ulam stability for various equations. A new type of stability for functional equations was introduced by T. Aoki [2] and Th.M. Rassias [20] by replacing ε in the Hyers theorem with a function depending on x and y, such that the Cauchy difference can be unbounded. For other results on the Hyers-Ulam stability of functional equations one can refer to [5, 15, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. The Hyers-Ulam stability of linear operators was considered for the first time in the papers by Miura, Takahasi et al. (see [7, 8, 14]). Similar type of results are obtained in [22] for weighted composition operators on $C(X)$, where X is a compact Hausdorff space. A result on the stability of a linear composition operator of the second order was given by J. Brzdek and S.M. Jung in [4].

Recently, Popa and Raşa obtained [17] a result on Hyers-Ulam stability of the Bernstein-Schnabl operators using a new approach to the Fréchet functional equation, and in [18, 19], they have shown the (in)stability of some classical operators defined on [0, 1] and found the best constant for the positive linear operators in the sense of Hyers-Ulam.

Motivated by their work, in this paper, we show the (in)stability of some complex positive linear operators on compact disk in the sense of Hyers-Ulam. We find the infimum of the Hyers-Ulam stability constants for complex Bernstein-Schurer operators and complex Kantrovich-Schurer operators on compact disk. Further we show that Lorentz polynomials are unstable in the sense of Hyers-Ulam on a compact disk.

2. The Hyers-Ulam stability property of operators

In this section, we recall some basic definitions and results on Hyers-Ulam stability property which form the background of our main results.

Definition 2.1. Let A and B be normed spaces and T a mapping from A into B. We say that T has the *Hyers-Ulam stability property* (briefly, T is *HU-stable*) [22] if there exists a constant K such that:
(i) for any $g \in T(A)$, $\varepsilon > 0$ and $f \in A$ with $\|Tf - g\| \leq \varepsilon$, there exists an $f_0 \in A$ such that $Tf_0 = g$ and $\|f - f_0\| \leq K\varepsilon$. The number K is called a *HUS constant of T*, and the infimum of all HUS constants of T is denoted by K_T. Generally, K_T is not a HUS constant of T ; see [7, 8].

Let now T be a bounded linear operator with the kernel denoted by $N(T)$ and the range denoted by $R(T)$. Consider the one-to-one operator $\widetilde{T}$ from the quotient space $A/N(T)$ into B:

$$\widetilde{T}(f + N(T)) = Tf, \quad f \in A,$$

and the inverse operator $\widetilde{T}^{-1} : R(T) \rightarrow A/N(T)$.

Theorem 2.2.([22]). *Let A and B be Banach spaces and $T : A \rightarrow B$ be a bounded linear operator. Then the following statements are equivalent:*
(a) *T is HU-stable;*
(b) *$R(T)$ is closed;*
(c) *$\widetilde{T}^{-1}$ is bounded.*
Moreover, if one of the conditions (a), (b), (c) *is satisfied, then $K_T = \|\widetilde{T}^{-1}\|$.*

Remark 2.3. (1) Condition (i) expresses the Hyers-Ulam stability of the equation
$Tf = g$, where $g \in R(T)$ is given and $f \in A$ is unknown.
(2) If $T : A \rightarrow B$ is a bounded linear operator, then (i) is equivalent to:
(ii) for any $f \in A$ with $\|Tf\| \leq 1$ there exists an $f_0 \in N(T)$ such that $\|f - f_0\| \leq K$, (see [13]).
So, in what follows, we shall study the HU-stability of a bounded linear operator $T : A \rightarrow B$ by checking the existence of a constant K for which (ii) is satisfied, or equivalently, by checking the boundedness of $\widetilde{T}^{-1}$.

The main results used in our approach for obtaining, in some concrete cases, the explicit value of K_T are the formula given above and a result by Lubinsky and Ziegler [12] concerning coefficient bounds in the Lorentz representation of a polynomial. Let $p \in \Pi_n$, where Π_n is the set of all polynomials of degree at most n with real coefficients. Then p has a unique Lorentz representation of the form

$$p(x) = \sum_{k=0}^{n} c_k x^k (1 - x)^{n-k}, \tag{2.1}$$

where $c_k \in \mathbb{R}$, $k = 0, 1, \cdots, n$. Remark that, in fact, it is a representation in Bernstein-Bézier basis. Let T_n denote the usual nth degree Chebyshev polynomial of the first kind. Then the following representation holds (see [12]):

$$T_n(2x - 1) = \sum_{k=0}^{n} d_{n,k} x^k (1 - x)^{n-k} (-1)^{n-k}, \tag{2.2}$$

where
$$d_{n,k} := \sum_{j=0}^{\min\{k,n-k\}} \binom{n}{2j} \binom{n-2j}{k-j} 4^j, \quad k = 0, 1, \cdots, n.$$

It is proved in [18] that $d_{n,k} = \binom{2n}{2k}$, $k = 0, 1, \cdots, n$. Therefore

$$T_n(2x-1) = \sum_{k=0}^{n} \binom{2n}{2k}(-1)^{n-k} x^k (1-x)^{n-k}.$$

Theorem 2.4.(*Lubinsky and Ziegler [12]*). *Let p have the representation (2.1), and let $0 \leq k \leq n$. Then*

$$|c_k| \leq d_{n,k}.\|p\|_\infty$$

with equality if and only if p is a constant multiple of $T_n(2x-1)$ where $\|p\|_\infty = \max\limits_{x \in [a,b]} |P(x)|$.

Let $C[0,1]$ be the space of all continuous, real-valued functions defined on $[0,1]$, and $C_b[0,+\infty)$ the space of all continuous, bounded, real-valued functions on $[0,+\infty)$. Endowed with the supremum norm, they are Banach spaces.

Popa and Raşa have shown the Hyers-Ulam stability of the following operators:

(i) Bernstein operators ([18])

For each integer $n \geq 1$, the sequence of classical Bernstein operators $B_n : C[0,1] \to C[0,1]$ is defined by (see [1])

$$B_n f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad f \in C[0,1], \ n \geq 1.$$

It is stable in the Hyers-Ulam sense and the best Hyers-Ulam stability constant is given by

$$K_{B_n} = \binom{2n}{2[\frac{n}{2}]}, n \in \mathbb{N}.$$

(ii) Szász-Mirakjan operators ([18])

The nth Szász-Mirakjan operator $L_n : C_b[0,+\infty) \to C_b[0,+\infty)$ defined by (see [1], pp. 338)

$$L_n f(x) = e^{-nx} \sum_{j=0}^{\infty} f\left(\frac{j}{n}\right) \frac{n^j}{j!} x^j, \quad x \in [0,+\infty)$$

is not stable in the sense of Hyers and Ulam for each $n \geq 1$.

(iii) Beta operators ([18])

For each $n \geq 1$, the Beta operator $B_n : C[0,1] \to C[0,1]$ defined by [13]

$$L_n f(x) := \frac{\int_0^1 t^{nx}(1-t)^{n(1-x)} f(t) dt}{\int_0^1 t^{nx}(1-t)^{n(1-x)} dt}$$

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is not stable in the sense of Hyers and Ulam.

(iv) Stancu operators ([19])

Let $C[0,1]$ be the linear space of all continuous functions $f : [0,1] \to \mathbb{R}$, endowed with the supremum norm denoted by $\|.\|$, and a, b real numbers, $0 \le a \le b$. The Stancu operator [21] $S_n : C[0,1] \to \Pi_n$ is defined by

$$S_n f(x) = \sum_{k=0}^{n} f\left(\frac{k+a}{n+b}\right) \binom{n}{k} x^k (1-x)^{n-k},$$

$f \in C[0,1]$. It is HU-stable and the infimum of the Hyers-Ulam constant is $K_{S_n} = \binom{2n}{2[\frac{n}{2}]} / \binom{n}{[\frac{n}{2}]}$, for each $n \ge 1$.

(v) Kantorovich operators ([19])

Let $X = \{f; \ f : [0,1] \to \mathbb{R},$ where f is bounded and Riemann integrable$\}$ be endowed with the supremum norm denoted by $\|.\|$. The Kantorovich operator defined by

$$K_n f(x) = (n+1) \sum_{k=0}^{n} \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt \right) \binom{n}{k} x^k (1-x)^{n-k},$$

$f \in X, x \in [0,1]$ is stable in Hyers-Ulam sense and the best HUS constant is $K_{S_n} = \binom{2n}{2[\frac{n}{2}]} / \binom{n}{[\frac{n}{2}]}$.

3. Main Results

In this section, we show the Hyers-Ulam stability of some other operators. Let D_R denote the compact disk having radius R, i.e., $D_R = \{z \in \mathbb{C} : |z| \le R\}$.

(i) Bernstein-Schurer Operators

Let $X_{D_R} = \{f : D_R \to \mathbb{C}$ be analytic in $D_R\}$ be the collection of all analytic functions endowed with the supremum norm denoted by $\|.\|$. The complex Bernstein-Schurer operator ([3]) $S_{n,p} : X_{D_R} \to \Pi_{n+p}$ is defined by

$$S_{n,p}(f)(z) = \sum_{k=0}^{n+p} \binom{n+p}{k} z^k (1-z)^{n+p-k} f\left(\frac{k}{n}\right), \quad z \in \mathbb{C}, \ f \in X_{D_R}.$$

We have $N(S_{n,p}) = \{f \in X_{D_R} : f(\frac{k}{n}) = 0, \ 0 \le k \le n+p\}$, which is a closed subspace of X_{D_R}, and $R(S_{n,p}) = \Pi_{n+p}$. The operator $\widetilde{S}_{n,p} : X_{D_R}/N(S_{n,p}) \to \Pi_{n+p}$ is bijective, $\widetilde{S}_{n,p}^{-1} : \Pi_{n+p} \to X_{D_R}/N(S_{n,p})$ is bounded since $\dim \Pi_{n+p} = 2(n+p+1)$. So according to Theorem 2.2 the operator $S_{n,p}$ is Hyers-Ulam stable.

Theorem 3.1. For $n \ge 1$

$$K_{S_{n,p}} = \|\widetilde{S}_{n,p}^{-1}\| = \binom{2(n+p)}{2[\frac{n+p}{2}]} / \binom{n+p}{[\frac{n+p}{2}]}.$$

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Proof. Let $q \in \Pi_{n+p}$, $\|q\| \leq 1$, and its Lorentz representation

$$q(z) = \sum_{k=0}^{n+p} c_k(q) z^k (1-z)^{n+p-k}, \quad |z| \leq R.$$

Consider the constant function $f_q \in X_{D_R}$ defined by

$$f_q\left(\frac{k}{n}\right) = \frac{c_k(q)}{\binom{n+p}{k}}, \quad 0 \leq k \leq n+p.$$

Then $S_{n,p} f_q = q$ and $\widetilde{S}_{n,p}^{-1}(q) = f_q + N(S_{n,p})$.

As usual, the norm of $\widetilde{S}_{n,p}^{-1} : \Pi_{n+p} \to X_{D_R}/N(S_{n,p})$ is defined by

$$\|\widetilde{S}_{n,p}^{-1}\| = \sup_{\|q\| \leq 1} \|\widetilde{S}_{n,p}^{-1}(q)\| = \sup_{\|q\| \leq 1} \inf_{h \in N(S_{n,p})} \|f_q + h\|.$$

Clearly

$$\inf_{h \in N(S_{n,p})} \|f_q + h\| = \|f_q\| = \max_{0 \leq k \leq n+p} |c_k(q)| / \binom{n+p}{k}.$$

Therefore

$$\|\widetilde{S}_{n,p}^{-1}\| = \sup_{\|q\| \leq 1} \max_{0 \leq k \leq n+p} |c_k(q)| / \binom{n+p}{k}$$

$$\leq \sup_{\|q\| \leq 1} \max_{0 \leq k \leq n+p} d_{n+p,k} \cdot \|q\| / \binom{n+p}{k} = \max_{0 \leq k \leq n+p} d_{n+p,k} / \binom{n+p}{k}.$$

On the other hand, let $r(z) = T_n(2z-1)$, $|z| \leq R$. Then $\|r\| = 1$ and $|c_k(r)| = d_{n+p,k}$, $0 \leq k \leq n+p$, according to Theorem 2.4. Consequently

$$\|\widetilde{S}_{n,p}^{-1}\| \geq \max_{0 \leq k \leq n+p} |c_k(r)| / \binom{n+p}{k} = \max_{0 \leq k \leq n+p} d_{n+p,k} / \binom{n+p}{k}$$

and so

$$\|\widetilde{S}_{n,p}^{-1}\| = \max_{0 \leq k \leq n+p} \frac{d_{n+p,k}}{\binom{n+p}{k}} = \max_{0 \leq k \leq n+p} \frac{\binom{2(n+p)}{2k}}{\binom{n+p}{k}}.$$

Let

$$a_k = \frac{\binom{2(n+p)}{2k}}{\binom{n+p}{k}}, \quad 0 \leq k \leq n+p.$$

Then

$$\frac{a_{k+1}}{a_k} = \frac{2n + 2p - 2k - 1}{2k + 1}, \quad 0 \leq k \leq n+p.$$

The inequality $\frac{a_{k+1}}{a_k} \geq 1$ is satisfied if and only if $k \leq [\frac{n+p-1}{2}]$, therefore

$$\max_{0 \leq k \leq n+p} a_k = a_{[\frac{n+p-1}{2}]+1} = \begin{cases} a_{[\frac{n+p}{2}]}, & n+p \text{ is even;} \\ a_{[\frac{n+p}{2}]+1}, & n+p \text{ is odd.} \end{cases}$$

Since $a_{[\frac{n+p}{2}]+1} a_{[\frac{n+p}{2}]}$ if $n+p$ is an odd number, we conclude that

$$K_{S_{n,p}} = \|\widetilde{S}_{n,p}^{-1}\| = \binom{2(n+p)}{2[\frac{n+p}{2}]} / \binom{n+p}{[\frac{n+p}{2}]}.$$

This completes the proof of the theorem.

(ii) Kantrovich-Schurer Operators

Let $X_{D_R} = \{f : D_R \to \mathbb{C}$ be analytic in $D_R\}$ be the collection of all analytic functions endowed with the supremum norm denoted by $\|.\|$. The complex Kantrovich-Schurer operator ([3])
   $L_{n,p} : X_{D_R} \to \Pi_{n+p}$ is defined by

$$L_{n,p}(f)(z) = (n+p+1) \sum_{k=0}^{n+p} \binom{n+p}{k} z^k (1-z)^{n+p-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \quad z \in \mathbb{C}, \ f \in X_{D_R}.$$

We have

$$N(L_{n,p}) = \{f \in X_{D_R} : \ f(t) = 0, \ t \in D_R\}.$$

The operators $L_{n,p}$ are Hyers-Ulam stable since their ranges are finite dimensional spaces.

Theorem 3.2. For $n \geq 1$

$$K_{L_{n,p}} = \|\widetilde{T}_{n,p}^{-1}\| = \frac{(n+1)\binom{2(n+p)}{2[\frac{n+p}{2}]}}{(n+p+1)\binom{n+p}{[\frac{n+p}{2}]}}.$$

Proof. Let $q \in \Pi_{n+p}$, $\|q\| \leq 1$, and its Lorentz representation

$$q(z) = \sum_{k=0}^{n+p} c_k(q) z^k (1-z)^{n+p-k}, \quad |z| \leq R.$$

Consider the constant function $f_q \in X_{D_R}$ defined by

$$f_q(t) = \frac{(n+1)c_k(q)}{(n+p+1)\binom{n+p}{k}}, \quad 0 \leq k \leq n+p, \ t \in D_R.$$

Then $L_{n,p}f_q = q$ and $\widetilde{L}_{n,p}^{-1}(q) = f_q + N(L_{n,p})$.
As usual, the norm of $\widetilde{L}_{n,p}^{-1} : \Pi_{n+p} \to X_{D_R}/N(L_{n,p})$ is defined by

$$\|\widetilde{L}_{n,p}^{-1}\| = \sup_{\|q\| \leq 1} \|\widetilde{L}_{n,p}^{-1}(q)\| = \sup_{\|q\| \leq 1} \inf_{h \in N(L_{n,p})} \|f_q + h\|.$$

Clearly

$$\inf_{h \in N(L_{n,p})} \|f_q + h\| = \|f_q\| = \max_{0 \leq k \leq n+p} \frac{(n+1)|c_k(q)|}{(n+p+1)\binom{n+p}{k}}.$$

Therefore

$$
\begin{aligned}
\|\widetilde{L}_{n,p}^{-1}\| &= \sup_{\|q\| \leq 1} \max_{0 \leq k \leq n+p} \frac{(n+1)|c_k(q)|}{(n+p+1)\binom{n+p}{k}} \\
&\leq \sup_{\|q\| \leq 1} \max_{0 \leq k \leq n+p} \frac{(n+1)\|q\|d_{n+p,k}}{(n+p+1)\binom{n+p}{k}} \\
&= \max_{0 \leq k \leq n+p} \frac{(n+1)d_{n+p,k}}{(n+p+1)\binom{n+p}{k}}.
\end{aligned}
$$

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On the other hand, let $r(z) = T_n(2z-1)$, $|z| \leq R$. Then $\|r\| = 1$ and $|c_k(r)| = d_{n+p,k}$, $0 \leq k \leq n+p$, according to Theorem 2.4. Consequently

$$\|\widetilde{L}_{n,p}^{-1}\| \geq \max_{0 \leq k \leq n+p} \frac{(n+1)|c_k(r)|}{(n+p+1)\binom{n+p}{k}} = \max_{0 \leq k \leq n+p} \frac{(n+1)d_{n+p,k}}{(n+p+1)\binom{n+p}{k}}$$

and so

$$\|\widetilde{L}_{n,p}^{-1}\| = \max_{0 \leq k \leq n+p} \frac{(n+1)d_{n+p,k}}{(n+p+1)\binom{n+p}{k}} = \max_{0 \leq k \leq n+p} \frac{(n+1)\binom{2(n+p)}{2k}}{(n+p+1)\binom{n+p}{k}}.$$

Let

$$a_k = \frac{(n+1)\binom{2(n+p)}{2k}}{(n+p+1)\binom{n+p}{k}}, \quad 0 \leq k \leq n+p.$$

Then

$$\frac{a_{k+1}}{a_k} = \frac{2n+2p-2k-1}{2k+1}, \quad 0 \leq k \leq n+p.$$

The inequality $\frac{a_{k+1}}{a_k} \geq 1$ is satisfied if and only if $k \leq [\frac{n+p-1}{2}]$, therefore

$$\max_{0 \leq k \leq n+p} a_k = a_{[\frac{n+p-1}{2}]+1} = \begin{cases} a_{[\frac{n+p}{2}]}, & \text{n+p is even;} \\ a_{[\frac{n+p}{2}]+1}, & \text{n+p is odd.} \end{cases}$$

Since $a_{[\frac{n+p}{2}]+1} = a_{[\frac{n+p}{2}]}$ if $n+p$ is an odd number, we conclude that

$$K_{L_{n,p}} = \|\widetilde{L}_{n,p}^{-1}\| = \frac{(n+1)\binom{2(n+p)}{2[\frac{n+p}{2}]}}{(n+p+1)\binom{n+p}{[\frac{n+p}{2}]}}.$$

This completes the proof of the theorem.

(iii) Lorentz Operators

The complex Lorentz polynomial [6] attached to any analytic function f in a domain containing the origin is given by

$$L_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{z}{n}\right)^k f^{(k)}(0), \quad n \in \mathbb{N}.$$

For $R > 1$ and denoting $D_R = \{z \in \mathbb{C}; |z| < R\}$, suppose that $f : D_R \to \mathbb{C}$ is analytic in D_R, i.e., $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$.

Theorem 3.3. For each $n \geq 1$, the Lorentz polynomial on compact disk is Hyers-Ulam unstable.

Proof. Let us denote $e_j(z) = z^j$, then from Lorentz operators we can easily obtain that $L_n(e_0)(z) = 1$, $L_n(e_1)(z) = e_1(z)$ and that for all $j, n \in \mathbb{N}$, $j \geq 2$, we have

$$
\begin{aligned}
L_n(e_j)(z) &= \binom{n}{j} j! \frac{z^j}{n^j}, \qquad 1 \leq R_1 < R \\
&= z^j \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{j-1}{n}\right).
\end{aligned}
$$

Also, since an easy computation shows that

$$L_n(f)(z) = \sum_{j=0}^{\infty} c_j L_n(e_j)(z), \quad \forall |z| \leq R_1,$$

and $L_n(e_0)(z) = 1$, $L_n(e_1)(z) = e_1(z)$. It follows that for each $j \geq 2$, $(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{j-1}{n})$ is an eigen value of L_n. It can be easily seen that L_n is injective. Therefore $1/(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{j-1}{n})$ is an eigen value of L_n^{-1}. Since

$$\lim_{j\to\infty} \frac{1}{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{j-1}{n})} = \lim_{j\to\infty} \frac{n^j}{(n-1)(n-2)\cdots(n-j+1)} = +\infty,$$

we conclude that L_n^{-1} is unbounded and so L_n is HU-unstable.

This completes the proof of the theorem.

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